2017 IEEE 56th Annual Conference on Decision and Control (CDC)
December 12-15, 2017, Melbourne, Australia

Edge-event-based multi-agent consensus with Zeno-free triggers under synchronized/unsynchronized clocks

Pengfei Fang, Qingchen Liu*, Xiaolei Hou, Jiahu Qin and Changbin Yu

Abstract— **In this paper, we present novel edge-event-based triggering algorithms to achieve multi-agent consensus with Zeno-free triggers. The control inputs and triggering conditions for each agent are designed based only on relative state measurements in each agent's own local coordinate system. The existence of strictly positive minimal inter-event times guarantees the elimination of Zeno behaviour. Two cases, namely the synchronized clock case and the unsynchronized clock case, are studied. In the synchronized clock case, all agents are activated simultaneously to measure the relative state information over edge links under a global clock. Edge events are defined and their occurrences trigger the update of control inputs for the two agents sharing the link. We show that average consensus can be achieved with our proposed algorithm. In the unsynchronized clock case, each agent executes control algorithms under its own clock which is not synchronized with other agents' clocks. An edge event only triggers control input update for an individual agent. Detailed explanations and analysis are provided to show that all agents will reach consensus in a totally asynchronous manner. Numerical simulations are given to verify the effectiveness of the proposed algorithms.**

I. Introduction

In recent years, research on the multi-agent consensus problems [1] under practical constraints is gaining much attention due to its closed connection to industrial applications. By practical constraints, mainly three types are considered, namely agent dynamics constraints, actuator constraints and communication/sensing constraints [2]. Among them, actuator constraints and communication/sensing constraints arise from the scenarios where agents are equipped with digital devices (processors, actuators, sensors and wireless transmitters/receivers) with limited performance. To deal with these practical constraints, time-scheduled [3], [4] and event-scheduled [5], [6], [7] control schemes have been introduced. Compared to time-scheduled control, event-scheduled control is more favourable in multi-agent systems since it provides aperiodic event triggers for information broadcasting and controller updates, which can reduce the requirement of on-board resources significantly.

From the viewpoint of the controller update, there are *three* main event-triggered schemes, which are widely adopted among vast numbers of papers regarding event-triggered multi-agent consensus problem [5], [6], [8], [9], [7], [10]. The first scheme was proposed in [5] where each agent triggers events and broadcasts its event times to its neighbours; the controller for each agent *is updated both at its own event times as well as the event times of its neighbours*. The second scheme proposed in [6] and developed in [11], [12] requires each agent to continuously measure the relative information over its edge links; by using combined relative measurements to design trigger condition, each agent only needs to *update its control input at its own event times*. The third scheme proposed in [7], [10] was also termed the edge-event-based triggering scheme. Trigger events are defined over each edge link and activate *the controller updates for two linked agents simultaneously*.

Note that two issues are not well addressed in all of the above mentioned work: 1) the discussion of global or local coordinate frames for information sensing, and 2) the assumption that all agents use synchronized clocks. In [13], the authors provide explanations about the coordinate frame requirements for [5], [8], [9], [6], [11]. However, this issue is not addressed in papers [7], [10] using edge-event-based schemes. On the other hand, the assumption of synchronized clocks is actually not reflective of many practical applications (e.g. robots are not likely be activated simultaneously). Achieving clock synchronization is a challenging task [14], [15]. We emphasize that the assumption of synchronized clocks plays a very important role in edge-event-based trigger scheme [7], [10], [16]. This is because two agents linked by one edge cannot trigger events simultaneously if they do not work under synchronized clocks i.e. synchronous controller updates for two linked agents cannot be guaranteed.

In this paper, we present novel edge-event-based algorithms to achieve multi-agent consensus with Zeno-free triggers under both synchronized and unsynchronized clocks. The agent's dynamics are modelled by single integrators and the graph topology is assumed to be fixed, undirected and connected. The contributions of this paper is two-fold. Firstly, as compared to [7], [10], the synchronized clock case studied in Section III provides another point of view with much simpler trigger conditions. In our framework, agents only use relative information measured in its own

* Corresponding author: qingchen.liu@anu.edu.au.

The work of P. Fang, Q. Liu, and C. Yu was supported by the Australian Research Council (ARC) under grants DP-130103610 and DP-160104500, by the National Natural Science Foundation of China (Grant 61375072), and by the Westlake Education Foundation. Q. Liu was supported by a China Scholarship Council Scholarship. The work of X. Hou was supported by Northwestern Polytechnical Univeristy through "the Fundamental Research Funds for the Central Universities" (3102016OQD052) and the National Natural Science Foundation of China (Grant 61703343). The work of J. Qin was supported by the National Natural Science Foundation of China (Grant 61473269).

P. Fang, Q. Liu and C. Yu are with Institute of Advanced Technology, Westlake Institute for Advanced Study, Zhejiang, China and Westlake University, Hangzhou, Zhejiang, China; Q. Liu is also with the Research School of Engineering, Australian National University, Canberra ACT 0200, Australia. X. Hou is with School of Automation, Northwestern Polytechnical Univeristy, Xi'an, 710129, China. J. Qin is with the Department of Automation, University of Science and Technology of China, Hefei 230027, China.

local coordinate frame to achieve *average consensus*. This is in contrast to prior work [17], [9] (a global coordinate frame is required for all agents) and [6], [11] (average consensus cannot be achieved). We also apply the time regulation idea from [11] to guarantee Zeno-free triggers, which differs from the time-dependent trigger condition used in [16]. Secondly, the unsynchronized clock case studied in Section IV provides a generalised framework for edge-event-based triggering schemes. The case involving synchronized clocks thus can be regarded as a special case. In this generalised framework, each agent measures the relative information and updates the control input under its own isolated clock. Edge events are defined over an individual agent rather than two linked agents, i.e. two agents linked by one edge do not update their control inputs synchronously. To the authors' knowledge, similar results are not found in the literature.

The rest of this paper is structured as follows. Section II provides mathematical notations and background on graph theory. In Section III and IV, the synchronized clock case and the unsynchronized clock case are detailed, respectively. Numerical simulations are provided at the end of the each section to verify the effectiveness of the proposed strategies. Finally, Section V concludes this paper and discusses a future research topic.

II. Preliminaries and Background

A. Notations

In this section, some basic notations are introduced. Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^n$ denote the natural number set, real number set and the n-dimensional real Euclidean space, respectively. The set of $m \times n$ real matrices is denoted by $\mathbb{R}^{m\times n}$. The empty set is denoted by $\emptyset$. The transpose of a vector or matrix M is denoted by M^T. $\lambda_i(M)$ denotes the i-th smallest eigenvalue of a symmetric matrix M. The Euclidean norm of a vector, and the matrix norm induced by the Euclidean norm, is denoted by $\|\cdot\|$.

B. Graph theory

A group of n agents is modelled by an undirected graph $\mathcal{G}$ with vertex set $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ and edge set $\mathcal{E} = \{\epsilon_1, \epsilon_2, \dots, \epsilon_m\} \subset \mathcal{V} \times \mathcal{V}$. A path in graph $\mathcal{G}$ from vertex v_{i_1} to vertex v_{i_j} is a sequence of distinct vertices starting from v_{i_1} and ending with v_{i_j} such that $(v_{i_k}, v_{i_{k+1}}) \in \mathcal{E}$ for $k = 1, 2, \dots, j-1$. A graph is called connected if there is a path between any two vertices. N_i, the neighbour set of node v_i, is defined as $N_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$. The adjacency matrix $A \in \mathbb{R}^{n\times n}$ of graph $\mathcal{G}$ indicates the vertex adjacency relationship, with entries $a_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. Let D be the $n \times n$ diagonal matrix of d_i's, where the degree d_i of each vertex i is given by $d_i = \sum_{j=1}^{n} a_{ij}$. The Laplacian matrix of G is a symmetric positive semi-definite matrix given by $L = D - A$. For a connected graph, the eigenvalues of L are denoted by $0 = \lambda_1(L) < \lambda_2(L) \leq \ldots \leq \lambda_n(L)$

Label the m edges from 1 to m and each edge is assigned an arbitrary orientation. Each entry of the $m \times n$ incidence matrix H of graph $\mathcal{G}$ are defined as $h_{ra} = 1$ (-1) if node v_a is the terminal (initial) node of r-th edge and $h_{ra} = 0$ otherwise. The incidence matrix H can be divided into two sub matrices: the in-incidence matrix $H_{\bigodot}$ and the out-incidence matrix $H_{\bigotimes}$. Following the definitions in [18], each entry of the $m \times n$ in-incidence matrix $H_{\bigodot}$ is denoted as $(h_{\bigodot})_{ra} = 1$ if node v_a is the terminal node of r-th edge and $(h_{\bigodot})_{ra} = 0$ otherwise. Similarly, each entry of the $m \times n$ out-incidence matrix $H_{\bigotimes}$ is denoted as $(h_{\bigotimes})_{ra} = -1$ if node v_a is the initial node of r-th edge and $(h_{\bigotimes})_{ra} = 0$ otherwise. It is obvious that $H = H_{\bigodot} + H_{\bigotimes}$.

Let $x_i \in \mathbb{R}$ denote a state that is assigned to agent i. The stack state vector $x = [x_1, x_2, \cdots, x_n]^T \in \mathbb{R}^n$ records all agents' states. It is well known that the relative state vector can be constructed as:

$$z = Hx, \tag{1}$$

where $z = [z_1, z_2, \cdots, z_m]^T \in \mathbb{R}^m$, with $z_r \in \mathbb{R}$ being the relative state over ϵ_r.

For an undirected graph, we have the following lemma:

Lemma 1: [19] If graph $\mathcal{G}$ is undirected and connected, then $z^T H H^T z \geq \lambda_2(L)\|z\|^2$, where $\lambda_2(L)$ refers to the smallest positive eigenvalue of Laplacian matrix L.

III. Synchronized clock case

A. Problem formulation

We assume that each agent is only equipped with relative position sensors, e.g. sonar or ToF (time-of-flight) camera, to measure the relative states between its neighbours and itself, in its own local coordinate frame. We further assume that all agents share a global clock t, i.e. each agent in the multi-agent system (MAS) is activated simultaneously. The sensing topology is captured by a fixed, undirected and connected graph $\mathcal{G}$ with corresponding incidence matrix H, Laplacian matrix L and adjacency matrix A. For each edge ϵ_r connecting agents i and j, both agents i and j measure the relative state z_r continuously.

The MAS we study in this paper consists of n single integrators labelled from 1 to n. The n agents are connected by m edges (sensing links), labelled from 1 to m. Let $x_i(t) \in \mathbb{R}$ denote the state of agent i, $i = 1, 2, \ldots n$. The dynamics of agent i are described by

$$\dot{x}_i(t) = u_i(t), \;\; i = 1, 2, \ldots, n, \tag{2}$$

where $u_i(t)$ is the control input. The sequence of event-triggered executions for edge ϵ_r is $t_{0_r} = 0, t_{1_r}, \ldots, t_{k_r}, \ldots$. At t_{k_r}, agent i and agent j linked by edge ϵ_r update their control inputs simultaneously. This synchronous controller updating phenomenon results from the fact that agents i and j share a global clock. We will provide detailed explanations about this phenomenon in the main result subsection. For agent i, which is one agent of the agent pair (i, j) linked by edge ϵ_r, the control input is designed as follows:

$$u_i(t) = \sum_{j\in N_i} (x_j(t_{k_r}) - x_i(t_{k_r})) \tag{3}$$

for $t \in [t_{k_r}, t_{k_r+1})$. We emphasize that only partial information $x_j(t_{k_r}) - x_i(t_{k_r})$ in the control input is updated at t_{k_r}.

Moreover, by observing (3), we see that control input uses only relative information.

The key problem in event-triggered control is to determine the next trigger time t_{k_r+1}, which ensures to achieve the consensus objective. In this section, we will design a novel edge-event-based algorithm with Zeno-free triggers for each agent.

B. Main result

We first introduce a time-varying error $e_r(t)$. For time $t \in [t_{k_r}, t_{k_r+1})$, the relative state measurement error for edge ϵ_r is defined as

$$e_r(t) = z_r(t_{k_r}) - z_r(t), \;\; r = 1, \ldots, m. \tag{4}$$

We note that $e_r(t)$ is actually calculated by agents i and j linked by ϵ_r separately using their own on-board processors. However, since agents i and j share a global clock, the values of $\|e_r(t)\|$ calculated inside their processors are identical. We then define the Zeno-free edge-event-based trigger algorithm by following the idea proposed in [11]. The next event time for edge ϵ_r is determined by

$$t_{k_r+1} = t_{k_r} + \max\{\tau_{k_r}, b_r\}, \tag{5}$$

where b_r is strictly positive and τ_{k_r} is determined by the trigger condition

$$f(e_r(t), z_r(t)) = \|e_r(t)\| - \beta_r \|z_r(t)\| = 0, \tag{6}$$

where $\beta_r > 0$. Every time an event is triggered, $e_r(t)$ resets to zero. Mathematically, τ_{k_r} is described by

$$\tau_{k_r} = \inf_{t > t_{k_r}} \{t - t_{k_r} | f(e_r(t), z_r(t)) = 0\}.$$

Theorem 1: Consider a multi-agent system where each agent's dynamics are described by (2) with control input (3) and the edge trigger condition (5). Let η_1 and η_2 be positive real numbers satisfying $\eta_1 + \eta_2 < 1$. If $\beta_r \leq \eta_1(\lambda_2(L)/\|H\|^2)$ for all edges, b_r is strictly positive and satisfies $b_r \leq \frac{\eta_2 \lambda_2(L)}{\|H\|^2(\sqrt{m}\|H\|^2 + \eta_2 \lambda_2(L))}$. Then

- (Average consensus) All agents' states converge to their initial average.
- (Zeno-free triggers) At any time $t > 0$, no edge will exhibit Zeno behaviour.

Proof: It is well-known that the compact form of continuous-time consensus dynamic is constructed $\dot{x} = -Lx = -H^T z$. Following this construction, the compact form of (3) can be written as

$$u(t) = -H^T \begin{bmatrix} z_1(t_{k_1}) \\ z_2(t_{k_2}) \\ \vdots \\ z_m(t_{k_m}) \end{bmatrix}, \tag{7}$$

where $k_r = \arg\max_{k_r \in \mathbb{N}} \{t_{k_r} | t_{k_r} \leq t\}, r = 1, \ldots, m$. By substituting the edge measurement error (4), the compact form of the consensus dynamic can be written as

$$\dot{x}(t) = -H^T z(t) - H^T e(t), \tag{8}$$

where $z(t) = [z_1(t), z_2(t), \ldots, z_m(t)]^T$ and $e(t) = [e_1(t), e_2(t), \ldots, e_m(t)]^T$.

Consider the following Lyapunov function $V(t) = \frac{1}{2} z(t)^T z(t)$. The time derivative of the Lyapunov function along (8) is

$$\dot{V}(t) = z(t)^T \dot{z}(t) = -z(t)^T H H^T z(t) - z(t)^T H H^T e(t).$$

From Lemma 1, we further obtain

$$\begin{aligned} \dot{V}(t) &\leq -\lambda_2(L)\|z(t)\|^2 + \|H\|^2 \|e(t)\| \|z(t)\| \\ &= -\left(\lambda_2(L) \sqrt{\sum_{r=1}^{m} \|z_r(t)\|^2} \right. \\ &\qquad \left. - \|H\|^2 \sqrt{\sum_{r=1}^{m} \|e_r(t)\|^2} \right) \|z(t)\|. \end{aligned}$$

If we can guarantee that

$$\sum_{r=1}^{m} \|e_r(t)\|^2 \leq \eta^2 \left(\frac{\lambda_2(L)}{\|H\|^2} \right)^2 \sum_{r=1}^{m} \|z_r(t)\|^2 \tag{9}$$

with $\eta \in (0, 1)$, then it yields

$$\dot{V}(t) \leq -\left((1-\eta)\lambda_2(L) \sqrt{\sum_{r=1}^{m} \|z_r(t)\|^2} \right) \|z(t)\| < 0. \tag{10}$$

According to (5), we know that at any time $t > 0$, the determination of inter-edge-event time of edge ϵ_r is either by τ_{k_r} or b_r. Let $S_1(t)$ and $S_2(t)$ be the edge sets consisting of edges whose next inter-edge-event time at t is τ_{k_r} and b_r, respectively. Then it is obvious that $S_1(t) \bigcup S_2(t) = \{\epsilon_1, \ldots, \epsilon_m\}$ and $S_1(t) \bigcap S_2(t) = \emptyset$. To guarantee (10), we further propose the following two conditions:

$$\sum_{\epsilon_r \in S_1(t)} \|e_r(t)\|^2 \leq \eta_1^2 \left(\frac{\lambda_2(L)}{\|H\|^2} \right)^2 \sum_{r=1}^{m} \|z_r(t)\|^2 \tag{11}$$

and

$$\sum_{\epsilon_r \in S_2(t)} \|e_r(t)\|^2 \leq \eta_2^2 \left(\frac{\lambda_2(L)}{\|H\|^2} \right)^2 \sum_{r=1}^{m} \|z_r(t)\|^2 \tag{12}$$

where η_1 and η_2 are strictly positive real numbers under the condition that $\eta_1 + \eta_2 = \eta < 1$. For each edge in $S_1(t)$, if we let $\beta_r \leq \eta_1(\lambda_2(L)/\|H\|^2)$, then condition (11) will holds for all t. For condition (12), if we can guarantee

$$\|e_r(t)\| \leq \sqrt{\zeta} \|z(t)\|, \tag{13}$$

where $\zeta = \frac{\eta_2^2}{m} \left(\frac{\lambda_2(L)}{\|H\|^2} \right)^2$, then condition (12) can be ensured. Since ζ is strictly positive, the evolution time of $\|e_r(t)\|/\|z(t)\|$ from 0 to $\sqrt{\zeta}$ is strictly positive (because $\|z(t)\| \neq 0$, $\|e_r(t)\|$ evolutes from 0 at t_{k_r}). By finding an upper bound B_r of this evolution time, we can determine a strictly positive time $b_r \leq B_r$. Then condition (13) can always be guaranteed if the evolution time of $\|e_r(t)\|/\|z(t)\|$

is b_r. To find B_r, we first estimate the time derivative of $\|e_r(t)\|/\|z(t)\|$:

$$\frac{d}{dt}\frac{\|e_r\|}{\|z\|} \leq \frac{\|\dot{e}_r\|}{\|z\|} + \frac{\|e_r\|}{\|z\|}\frac{\|\dot{z}\|}{\|z\|}. \tag{14}$$

According to (4), one can deduce that $\dot{e}_r = -\dot{z}_r$. So it is obvious that $\frac{d}{dt}\frac{\|e_r\|}{\|z\|} \leq \|H\|^2(1+\frac{\|e\|}{\|z\|})^2$. Similar time derivative of $\|e(t)\|/\|z(t)\|$ yields $\frac{d}{dt}\frac{\|e\|}{\|z\|} \leq \|H\|^2(1+\frac{\|e\|}{\|z\|})^2$. It is noticed that $\|e\|/\|z\|$ always upper bounds $\|e_r\|/\|z\|$ and both of them are non-negative. Now we conclude that $\|e_r\|/\|z\| < g(t, g_0)$, where $g(t, g_0)$ is the solution of $\dot{g}(t) = \|H\|^2(1+g(t))^2$, $g_0 = 0$. Thus the lower bound of evolution time of $\|e_r\|/\|z\|$ from 0 to $\sqrt{\zeta}$ is

$$B_r = \frac{\eta_2\lambda_2(L)}{\|H\|^2\left(\sqrt{m}\|H\|^2+\eta_2\lambda_2(L)\right)}. \tag{15}$$

We can choose a strictly positive real time b_r which is satisfied with $b_r \leq B_r$ to guarantee (12) for each edge in $S_2(t)$. Since b_r is strictly positive, it is straightforward to conclude that Zeno behaviour is excluded for each edge. Moreover, since condition (10) can be ensured, we also conclude that consensus can be reached.

The proof to show that the consensus value is the initial average is standard thus omitted from this paper. ■

C. Simulation

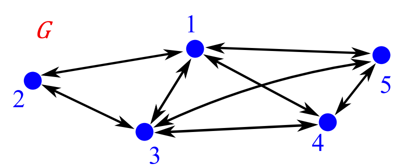

Fig. 1: Graph topology.

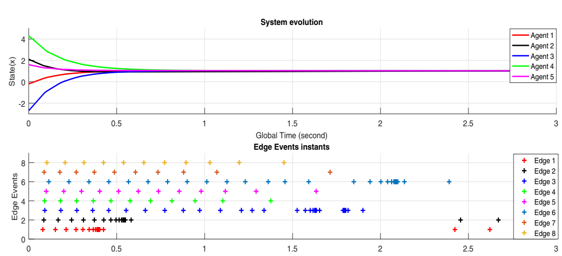

Fig. 2: State trajectories and edge event times

The MAS considered in the simulation consists of 5 agents. The sensing topology is described by Fig. 1. The initial states for all agents are set as $x_1(0) = -0.2$, $x_2(0) = 2.1$, $x_3(0) = -2.7$, $x_4(0) = 4.3$ and $x_5(0) = 1.6$. The parameters are chosen as $\eta_1 = 0.85$ and $\eta_2 = 0.14$, then we also set $\beta_r = 0.34$ for the trigger function (6). The time interval b_r is chosen as $b_r = 0.0039s$, which satisfies the condition (15).

The state trajectories and trigger performance for the Zeno-free algorithm are shown in Fig. 2. The minimum inter-edge-event time interval observed in the simulation is $0.0039s$, which corresponds to the value of b_r calculated according to (15).

IV. Unsynchronized Clock Case

A. Problem formulation

Let t, $t(0) = 0$ denote the global clock. However, each agent i has its own isolated, local clock t^i, $i = 1, 2, \ldots, n$. Let $t^i(0) \geq 0$ denote the initial value for each t^i and $t^i(0), \forall i$ is not necessarily identical. That is to say, agents i and j linked by edge ϵ_r start to measure the relative information and update their control inputs under their own clocks with non-identical initial time. The main challenge in this section arises from the fact that agent i and j linked by ϵ_r do not update their control inputs synchronously.

The MAS we study in this section also consists of n single integrators labelled from 1 to n. Let $x_i(t^i) \in \mathbb{R}$ denote the state of agent i, $i = 1, 2, \ldots n$. The dynamics of agent i are given as

$$\dot{x}_i(t^i) = u_i(t^i), \quad i = 1, 2, \ldots, n, \tag{16}$$

where $u_i(t^i)$ is control input.

Note that the trigger times of the agents i and j linked by ϵ_r are non-identical, we define two time sequences of event-triggered executions for agents i and j, respectively, which are $t^i_{0^i_r}, t^i_{1^i_r}, \ldots, t^i_{k^i_r}, \ldots$ for agent i under t^i and $t^j_{0^j_r}, t^j_{1^j_r}, \ldots, t^j_{k^j_r}, \ldots$ for agent j under t^j. $t^i_{k^i_r}$ denotes the time of k-th edge event of agent i triggered over edge ϵ_r under agent i's clock. Both agents update their control inputs at their own edge event times. For agent i, which is one agent of the agent pair (i, j) linked by ϵ_r, the control input is designed as follows:

$$u_i(t^i) = \sum_{j \in N_i}\left(x_j(t^i_{k^i_r}) - x_i(t^i_{k^i_r})\right), \quad i = 1, 2, \ldots, n \tag{17}$$

for $t^i \in [t^i_{k^i_r}, t^i_{k^i_r+1})$. In this section, we will aim to design a Zeno-free trigger scheme to determine the trigger times.

B. Main result

For time $t^i \in [t^i_{k^i_r}, t^i_{k^i_r+1})$, agent i, which is one agent of edge ϵ_r, measures the relative states $z^i_r(t^i)$ continuously along its own time axis and the relative state measurement error is defined as

$$e^i_r(t^i) = z^i_r(t^i_{k^i_r}) - z^i_r(t^i). \tag{18}$$

Since $t^i_{0^i_r} = t^j_{0^j_r}$ can not be guaranteed for agents i and j linked by ϵ_r, it is obvious that $e^i_r(t^i)$ is not supposed to be equal to $e^j_r(t^j)$. When combined the trigger conditions proposed below, it is implied that the linked agents i and j update their controllers asynchronously.

We follow the same method used in the Zeno-free algorithm of Section III to determine the next edge event time over ϵ_r for agent i:

$$t^i_{k^i_r+1} = t^i_{k^i_r} + \max\{\tau^i_{k^i_r}, b_r\}. \tag{19}$$

The trigger function used to determine $\tau^i_{k^i_r}$ is

$$f(e^i_r(t^i), z^i_r(t^i)) = \|e^i_r(t^i)\| - \beta^i_r\|z^i_r(t^i)\| \tag{20}$$

where $\beta^i_r > 0$. As usual, every time the trigger condition (19) is satisfied, $e^i_r(t^i)$ is reset to zero.

***Theorem** 2:* Consider system (16) with control input (17), trigger function (19). Let η_1 and η_2 be positive real numbers and $\eta_1 + \eta_2 < 1$. Let $\alpha = \max\{\|HH^T_{\bigotimes}\|, \|HH^T_{\bigodot}\|\}$. If $\beta^i_r \le \eta_1\lambda_2(L)/2\alpha$ for all edges, b_r is strictly positive and satisfies $b_r \le \frac{\eta_2\lambda_2(L)}{2\alpha(2m\alpha+\eta_2\lambda_2(L))}$. Then

- (Consensus) All agents' states will reach consensus.
- (Zeno-free triggers) No agent will exhibit Zeno behaviour.

Proof: It is obvious that we do not need to consider the convergence of the system before all agents are activated. Thus we introduce a new global clock t', where $t'(0) = \max\{t^i(0) : i = 1, 2, \ldots, n\}$ indicates the time point that all agents are activated to achieve consensus. Note that the compact form (7) cannot be used here because agents i and j linked by edge ϵ_r update asynchronously. New variables are required to be defined to construct the compact form of the system.

Note that all the state variables used and defined in the proof are with respect to a global coordinate frame. We start the analysis from the continuous-time consensus dynamic $\dot{x}(t') = -H^T z(t')$, as well. In this dynamic, the entry h^T_{ar} of H^T can be explained as follows:

$$h^T_{ar} = \begin{cases} 1, & \text{agent } a\text{'s knowledge of } z_r(t') \text{ is } -z_r(t') \\ -1, & \text{agent } a\text{'s knowledge of } z_r(t') \text{ is } z_r(t') \\ 0, & \text{agent } a \text{ does not access } z_r(t'). \end{cases} \tag{21}$$

Note that $h^T_{ar} = h_{ra}$, h_{ra} is the entry of H. According to the definition of h_{ra} in (21), we have the following conclusions: if agent a is the terminal agent of edge ϵ_r, its knowledge of $z_r(t')$ is $-z_r(t')$; if agent a is the initial agent of edge ϵ_r, its knowledge of z_r is $z_r(t')$. Let the relative states assigned to initial agent i and terminal agent j linked by edge ϵ_r be respectively described by z^μ_r and z^ν_r, where the initial agent and terminal agent are pre-assigned by incidence matrix H. It is obvious that $z^\mu_r(t') = z^\nu_r(t') = z_r(t')$. Note that there are m initial agents and m terminal agents in the MAS since the graph $\mathcal{G}$ has m edges. Then it is reasonable to rewrite the consensus dynamic as $\dot{x} = -H^T_{\bigotimes} z^\mu(t') - H^T_{\bigodot} z^\nu(t')$, where $z^\mu = [z^\mu_1, z^\mu_2, \ldots, z^\mu_m]^T$ and $z^\nu = [z^\nu_1, z^\nu_2, \ldots, z^\nu_m]^T$.

Let $t'_{k^\mu_r}$ and $t'_{k^\nu_r}$ re-denote the latest r-th edge event time instants of initial agent i and terminal agent j linked by edge ϵ_r, respectively. It is assumed that $t'_{k^\mu_r}, t'_{k^\nu_r} \ge t'(0)$. Following the consensus dynamic constructed in the last paragraph, the compact form of the control input (17) can be expressed as:

$$u(t') = -H^T_{\bigotimes} \begin{bmatrix} z^\mu_1(t'_{k^\mu_1}) \\ z^\mu_2(t'_{k^\mu_2}) \\ \vdots \\ z^\mu_m(t'_{k^\mu_m}) \end{bmatrix} - H^T_{\bigodot} \begin{bmatrix} z^\nu_1(t'_{k^\nu_1}) \\ z^\nu_2(t'_{k^\nu_2}) \\ \vdots \\ z^\nu_m(t'_{k^\nu_m}) \end{bmatrix}, \tag{22}$$

which is the key step in the proof.

According to (18), we define two stack measurement error vectors $e^\mu = [e^\mu_1, e^\mu_2, \ldots, e^\mu_m]^T$ and $e^\nu = [e^\nu_1, e^\nu_2, \ldots, e^\nu_m]^T$ are defined for all of the initial agents and terminal agents, respectively. The compact form of the consensus dynamic at t' can be formulated as

$$\begin{aligned}\dot{x}(t') &= -H^T_{\bigotimes} z(t') - H^T_{\bigodot} z(t') - H^T_{\bigotimes} e^\mu(t') - H^T_{\bigodot} e^\nu(t') \\ &= -H^T z(t') - H^T_{\bigotimes} e^\mu(t') - H^T_{\bigodot} e^\nu(t'). \end{aligned} \tag{23}$$

Now reconsider the Lyapunov function $V(t') = \frac{1}{2} z(t')^T z(t')$. Its time derivative along (23) is

$$\begin{aligned}\dot{V}(t') &= z(t')^T H \dot{x}(t') \\ &= -z(t')^T H H^T z(t') - z(t')^T H H^T_{\bigotimes} e^\mu(t') \\ &\quad - z(t')^T H H^T_{\bigodot} e^\nu(t').\end{aligned}$$

By recalling Lemma 1, it yields that

$$\begin{aligned}\dot{V}(t') &\le -\lambda_2(L)\|z(t')\|^2 + \|HH^T_{\bigotimes}\|\|z(t')\|\|e^\mu(t')\| \\ &\quad + \|HH^T_{\bigodot}\|\|z(t')\|\|e^\nu(t')\| \\ &= -\Big(\lambda_2(L)\|z(t')\| - \|HH^T_{\bigotimes}\|\|e^\mu(t')\| \\ &\quad -\|HH^T_{\bigodot}\|\|e^\nu(t')\|\Big)\|z(t')\|\end{aligned}$$

Note that $\|e^\mu(t')\| = \sqrt{\sum_{r=1}^m \|e^\mu_r(t')\|^2}$ and $\|e^\nu(t')\| = \sqrt{\sum_{r=1}^m \|e^\nu_r(t')\|^2}$. Let $\alpha = \max\{\|HH^T_{\bigotimes}\|, \|HH^T_{\bigodot}\|\}$. If we can ensure the following condition

$$\sqrt{\sum_{r=1}^m \|e^\mu_r(t')\|^2} + \sqrt{\sum_{r=1}^m \|e^\nu_r(t')\|^2} \le \frac{\lambda_2(L)}{\alpha}\|z(t')\|, \tag{24}$$

then consensus will be achieved.

At t', let $S^1_\mu(t')$ and $S^2_\mu(t')$ be the edge sets that their linked initial agents will trigger the edge events at $t'_{k^\mu_r} + \tau_{k^\mu_r}$ and $t'_{k^\mu_r} + b_r$, respectively. It is satisfied that $S^1_\mu(t') \bigcup S^2_\mu(t') = \{\epsilon_1, \ldots, \epsilon_m\}$ and $S^1_\mu(t') \bigcap S^2_\mu(t') = \emptyset$. Similarly, let $S^1_\nu(t')$ and $S^2_\nu(t')$ denote the edge sets that their linked terminal agents will trigger the edge events at $t'_{k^\mu_r} + \tau_{k^\mu_r}$ and $t'_{k^\mu_r} + b_r$, respectively. it is also satisfied that $S^1_\nu(t') \bigcup S^2_\nu(t') = \{\epsilon_1, \ldots, \epsilon_m\}$ and $S^1_\mu(t') \bigcap S^2_\mu(t') = \emptyset$. Note that condition (24) can be guaranteed if

$$\sqrt{\sum_{r\in S^1_\mu(t')} \|e^\mu_r(t')\|^2} + \sqrt{\sum_{r\in S^1_\nu(t')} \|e^\nu_r(t')\|^2} \le \frac{\eta_1\lambda_2(L)}{\alpha}\|z(t')\| \tag{25}$$

and

$$\sqrt{\sum_{r\in S^2_\mu(t')} \|e^\mu_r(t')\|^2} + \sqrt{\sum_{r\in S^2_\nu(t')} \|e^\nu_r(t')\|^2} \le \frac{\eta_2\lambda_2(L)}{\alpha}\|z(t')\| \tag{26}$$

where $\eta_1, \eta_2 > 0$ and $\eta_1 + \eta_2 < 1$.

According to the trigger function (20) and the fact that the measurement error (18) is reset as soon as the value of the trigger function reaches zero, it is enough to imply $\|e^\mu_r(t')\| \le \beta_{\max}\|z_r(t')\|$ and $\|e^\nu_r(t')\| \le \beta_{\max}\|z_r(t')\|$, where $\beta_{\max} = \max\{\beta^i_r\}$. Furthermore, by recalling that $S^1_\mu(t')$ and $S^1_\nu(t')$ are subsets of edge set $\mathcal{E}$, we obtain

$\mathbf{card}\{S^1_\mu(t')\}, \mathbf{card}\{S^1_\nu(t')\} \leq m$. The above analysis indicates that the upper bound of the left-hand side term in (25) can be calculated as $2\sqrt{\sum_{r=1}^{m} \beta^2_{\max}\|z_r(t')\|^2}$, which is equal to $2\beta_{\max}\|z(t')\|$. If we enforce β^i_r to satisfy $\beta^i_r < \frac{\eta_1\lambda_2(L)}{2\alpha}$, then condition (25) is always satisfied.

For condition (26), since $\mathbf{card}\{S^2_\mu(t')\} \leq m$, we obtain $\sqrt{\sum_{r\in S^2_\mu(t')} \|e^\mu_r(t')\|^2} \leq \sum_{r=1}^{m} \|e^\mu_r(t')\|$. According to the same arguments, we also get $\sqrt{\sum_{r\in S^2_\nu(t')} \|e^\mu_r(t')\|^2} \leq \sum_{r=1}^{m} \|e^\nu_r(t')\|$. The upper bound of the left-hand side term in (26) is thus obtained as $\sum_{r=1}^{m} \|e^\mu_r(t')\| + \sum_{r=1}^{m} \|e^\nu_r(t')\|$. Note that $e^\mu_r(t')$ and $e^\nu_r(t')$ are actually the measurement error $e^i_r(t')$ defined in (18). By enforcing $\|e^i_r(t')\| \leq \frac{\eta_2\lambda_2(L)}{2m\alpha}\|z(t')\|$, condition (26) can be ensured. Now we are ready to determine B_r. By following the similar process from (14) to (15) in the last subsection, the lower bound B_r is obtained as

$$B_r = \frac{\eta_2\lambda_2(L)}{2\alpha(2m\alpha + \eta_2\lambda_2(L))}. \tag{27}$$

For each agent i, the next edge-triggering time $t^i_{k^i_r+1}$ can be set as $t^i_{k^i_r} + b_r$, where $b_r \leq B_r$, if $\tau^i_{k^i_r}$ determined by trigger function (20) is less than b_r. By choosing suitable β_r and b_r, the aims of both consensus and Zeno-free triggers can be achieved. ■

C. Simulation

The sensing topology and the initial states of all agents are set to be the same with those in the simulation of Section III. Parameters are chosen as $\eta_1 = 0.8$ and $\eta_2 = 0.19$. Thus β^i_r and b_r are calculated as $\beta^i_r = 0.22$ and $b_r = 0.0011s$, respectively. The activated times are chosen as $t^1(0) = 0.4$, $t^2(0) = 0.7$, $t^3(0) = 0.1$, $t^4(0) = 0.2$ and $t^5(0) = 0.8$. Fig. 3 illustrates the trajectories of the MAS and the event times for both agent 1 and agent 2 linked by edge ϵ_1. We can see that consensus can be reached and the trigger times of agents 1 and 2 triggered by ϵ_1 are asynchronous.

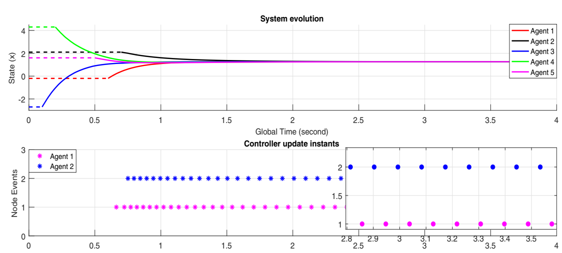

Fig. 3: State trajectories and edge event times both agents 1 and 2 triggered over edge ϵ_1

V. Conclusion

In this paper, we propose novel Zeno-free, edge-event-based algorithms to achieve multi-agent consensus under both synchronized clocks and unsynchronized clocks. In the synchronized clock case, we show that average consensus can be achieved under our algorithms even though each agent only measure the relative information via its own local coordinate frame. In the study of the unsynchronized clock case, each agent not only uses the relative information, but also works under its own clock that is not necessarily synchronized with others' clocks. We show that consensus can be achieved with Zeno-free triggers by using our proposed algorithm. Future work will focus on the relaxation of using global information of networks to design trigger conditions.

References

[1] W. Ren and R. W. Beard, *Distributed consensus in multi-vehicle cooperative control*. Springer, 2008.

[2] J. Qin, Q. Ma, Y. Shi, and L. Wang, "Recent advances in consensus of multi-agent systems: A brief survey," *Industrial Electronics, IEEE Transactions on*, vol. pp, pp. 1–1, December 2016.

[3] N. Huang, Z. Duan, and G. Chen, "Some necessary and sufficient conditions for consensus of second-order multi-agent systems with sampled position data," *Automatica*, vol. 63, no. 1, pp. 214 – 155, 2016.

[4] J. Qin and H. Gao, "A sufficient condition for convergence of sampled-data consensus for double-integrator dynamics with nonuniform and time-varying communication delays," *IEEE Transactions on Automatic Control*, vol. 57, no. 9, pp. 2417–2422, 2012.

[5] D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, "Distributed event-triggered control for multi-agent systems," *Automatic Control, IEEE Transactions on*, vol. 57, no. 5, pp. 1291 – 1297, 2012.

[6] Y. Fan, G. Feng, Y. Wang, and C. Song, "Distributed event-triggered control of multi-agent systems with combinational measurements," *Automatica*, vol. 49, no. 2, pp. 671 – 675, 2013.

[7] F. Xiao, X. Meng, and T. Chen, "Average sampled-data consensus driven by edge events," in *Chinese Control Conference, 2012. 31st IEEE Conference on*, pp. 6239 – 6244, IEEE, 2012.

[8] G. S. Seyboth, D. V. Dimarogonas, and K. H. Johansson, "Event-based broadcasting for multi-agent average consensus," *Automatica*, vol. 49, no. 1, pp. 245–252, 2013.

[9] C. Nowzari and J. Cortés, "Zeno-free, distributed event-triggered communication and control for multi-agent average consensus," in *American Control Conference (ACC), 2014*, pp. 2148–2153, IEEE, 2014.

[10] F. Xiao, X. Meng, and T. Chen, "Sampled-data consensus in switching networks of integrators based on edge events," *International Journal of Control*, vol. 88, no. 2, pp. 391 – 402, 2015.

[11] Y. Fan, L. Liu, G. Feng, and L. Wang, "Self-triggered consensus for multi-agent systemswith zeno-free triggers," *Automatic Control, IEEE Transactions on*, vol. 60, no. 10, pp. 2779 – 2784, 2015.

[12] Q. Liu, J. Qin, and C. Yu, "Event-based multi-agent cooperative control with quantized relative state measurements," in *Decision and Control, 2016. 55th IEEE Conference on*, pp. 2233 – 2239, IEEE, 2016.

[13] Z. Sun, N. Huang, B. D. O. Anderson, and Z. Duan, "A new distributed zeno-free event-triggered algorithm for multi-agent consensus," in *Decision and Control, 2016. 55th IEEE Conference on*, pp. 3444 – 3449, IEEE, 2016.

[14] R. Carli and S. Zampieri, "Networked clock synchronization based on second order linear consensus algorithms," in *Decision and Control (CDC), 2010 49th IEEE Conference on*, pp. 7259–7264, IEEE, 2010.

[15] G. S. Seyboth and F. Allgower, "Clock synchronization over directed graphs," in *Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on*, pp. 6105–6111, IEEE, 2013.

[16] B. Wei, F. Xiao, and M.-Z. Dai, "Edge event-triggered control for multi-agent systems under directed communication topologies," *International Journal of Control*, vol. 0, no. 0, pp. 1–10, 0.

[17] G. Seyboth, "Event-based control for multi-agent systems," *Master's Degree Project, Stockholm, Sweden*, 2010.

[18] Z. Zeng, X. Wang, and Z. Zheng, "Edge agreement of multi-agent system with quantised measurements via the directed edge laplacian," *Control Theory & Application, IET*, vol. 10, pp. 1583 – 1589, August 2016.

[19] M. Guo, "Quantized cooperative control," *Master's Degree Project, Stockholm, Sweden*, 2011.